International Journal of Theoretical Physics, Vol. 34, No. 3, 1995

Mass of a Body in Brans–Dicke Theory

Hassan Randjbar Askari[1] and Nematullah Riazi[1]

Received May 1, 1994

Exploiting exact spherical solutions of the Brans–Dicke equations, we study various definitions of the total mass of a body in this theory. We argue why the vacuum spherical solutions involve—in general—two arbitrary constants of integration. We discuss the dependence of the total mass on these constants.

1. INTRODUCTION

The equivalence of the inertial and passive gravitational masses of a body, which is supported by sensitive experiments (Weinberg, 1972), lies at the foundations of general relativity.

A body experiences a force proportional to its *passive* gravitational mass when put inside a background gravitational field. The same body produces a gravitational field which is asymptotically proportional to its *active* gravitational mass. On the other hand, when the body is acted on by, say, electromagnetic forces, it undergoes an acceleration which is inversely proportional to its *inertial* mass. The inertial force which appears in accelerating reference frames is also proportional to the body's *inertial* mass.

In Newtonian physics, all the various definitions of mass are taken to be identical, while in the general theory of relativity, the equivalence of passive and active gravitational masses is still disputed by some authors.

Bonnor (1992) recently claimed that the active and passive gravitational masses of a body can be different. He calculated these quantities for a static sphere whose structure is given by the Schwarzschild interior solution for a perfect fluid of uniform rest density. He concluded that the fractional differences between the active and passive gravitational masses for the sun, earth, and moon are $\sim 2 \times 10^{-6}$, $\sim 7 \times 10^{-10}$, and $\sim 3 \times 10^{-11}$, respectively.

[1]Physics Department and Biruni Observatory, Shiraz University, Shiraz 71454, Iran. Fax: (INT+98) 071 20027.

Bonnor used the coefficient of acceleration in the equation of motion of a perfect fluid and integrated this over the proper volume to obtain the inertial (or passive gravitational) mass.

Rosen and Cooperstock (1992) took into account the gravitational energy of a body. They argued against Bonnor and showed that the active and passive gravitational masses are equal, if we include this self-energy.

Subsequently, Herrera and Ibanẽz (1993) showed that for the Schwarzschild sphere, the difference between the active and passive gravitational masses as calculated by Bonnor is equal to the work required to build up the sphere. They concluded that the total energy of a body is the same as M_a, but is different from M_p as defined by Bonnor (with $\rho + p$ as the passive gravitational mass density).

In the Brans–Dicke theory of gravitation (Brans and Dicke, 1961), which was originally proposed to include Mach's principle, the problem of the mass of a body becomes more involved. In this paper, we discuss various definitions for the mass of a body. We exploit exact spherical solutions of the BD theory which are asymptotically flat and contain two arbitrary parameters (or constants of integration). We study the dependence of the total mass of the configuration on these two parameters.

2. ACTIVE AND PASSIVE GRAVITATIONAL MASSES IN GENERAL RELATIVITY

Far enough from an arbitrary material body, it is g_{00} in the metric coefficients which effectively determines the state of motion of a slowly moving test particle. In general relativity, the motion of a test particle is governed by the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{1}$$

Accordingly, we can determine the *active* mass of the central body by comparing g_{00} with $-(1 - 2GM_a/r)$ in the limit $r \to \infty$.

A more elegant formalism to find M_a is to define the energy-momentum pseudotensor of the gravitational field. The total energy (mass) of the combined gravitational field plus the material system is then (Weinberg, 1972)

$$M = P^0 = \int \tau^{00}\, d^3x = -\frac{1}{16\pi G}\int \left(\frac{\partial h_{jj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^j}\right) n_i r^2\, d\Omega \tag{2}$$

In this equation, $\tau^{\mu\nu} = t^{\mu\nu} + T^{\mu\nu}$ is the total (gravitational plus material) energy-momentum pseudotensor, and h_{ij} are the spacelike components of the infinitesimal deviations from the Minkowski metric tensor at large distances

$$g_{ij} = \eta_{ij} + h_{ij}, \qquad i, j = 1, 2, 3 \tag{3}$$

Note that in order to perform the integral (2) it is necessary to choose quasi-Euclidean coordinates such that

$$r = (x_i x_i)^{1/2}, \qquad n_i = \frac{x_i}{r}, \qquad \text{and} \qquad d\Omega = \sin\theta \, d\theta \, d\phi \tag{4}$$

We use Weinberg's notations and conventions throughout this paper. Both procedures described above lead to the same M_a for the Schwarzschild metric.

In order to define the passive gravitational mass, Bonnor (1992) employed the energy-momentum conservation equation $T^{\mu\nu}_{;\mu} = 0$, which for a perfect fluid becomes

$$(p + \rho)a^{\mu} = (g^{\mu\nu} - U^{\mu}U^{\nu}) \frac{\partial p}{\partial x^{\nu}} \tag{5}$$

In this equation ρ and p are the proper energy density and proper pressure, respectively, and U^{μ} is the unit 4-velocity. Bonnor interpreted $p + \rho$ as the inertial mass density of the fluid concerned. The inertial mass density is equal to the passive gravitational mass density according to the principle of equivalence. The passive mass of the body is therefore

$$M_p = \int (p + \rho)\sqrt{^3g}\, d^3x \tag{6}$$

Bonnor then calculated M_p for a Schwarzschild sphere of uniform rest density and obtained

$$M_p = M_a\left[1 + \frac{4M_a}{r_1} + O\left(\left(\frac{M_a}{r_1}\right)^2\right)\right] \tag{7}$$

where $r_1 = [3M_\alpha/4\pi\rho]^{1/3}$, and M_a/r_1 is assumed to be sufficiently small. He concluded that for a Schwarzschild sphere of sufficiently small M_a/r_1, the passive gravitational mass is greater than the active gravitational mass and

$$\frac{M_p - M_a}{M_a} \sim \frac{4}{5}\frac{M_a}{r_1} \tag{8}$$

Rosen and Cooperstock (1992) showed that the mass equivalent of the gravitational energy of the fluid element which was used by Bonnor to calculate M_p was not taken into account by him. They thus considered this contribution and used the Lorentz tensor

$$\Theta_{\mu}^{\nu} = \int \sqrt{g}\,(T_{\mu}^{\nu} + t_{\mu}^{\nu})\,d^3x \tag{9}$$

to derive the following expression for the passive (or inertial) mass of the body:

$$M_p = \int \sqrt{g}\left(\rho + p + t_0^0 - \frac{1}{3}\,t_k^k\right) d^3x \tag{10}$$

They then showed that in the case of a Schwarzschild sphere, this leads precisely to the same expression as that for the active gravitational mass. Also, Herrera and Ibanẽz (1993) showed that the difference between M_a and M_p as defined by Bonnor is equal to the work done in building the Schwarzschild sphere.

3. MASS OF A BODY IN BRANS–DICKE THEORY

Let us now turn to BD theory, and reexamine the mass of a body in the framework of this theory. The BD field equations, which are a generalization of GR, read

$$\Box^2\phi = \frac{8\pi}{3 + 2\omega}\,T_{M\mu}^{\mu}$$

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = -\frac{8\pi}{\phi}\,(T_{M\mu\nu} + T_{\phi\mu\nu}) \tag{11}$$

where

$$T_{\phi\mu\nu} = \frac{\omega}{8\pi\phi}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}\,g_{\mu\nu}\phi_{;\rho}\phi^{\rho}_{;}\right) + \frac{1}{8\pi}\,(\phi_{;\mu;\nu} - g_{\mu\nu}\Box^2\phi) \tag{12}$$

In these equations, ϕ is a real scalar field (the BD field), and ω is a dimensionless parameter. The Einstein field equations are the $\omega \rightarrow \infty$ limit of (11). Assuming the spherically symmetric, static metric

$$d\tau^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2\,d\theta^2 - r^2\sin^2(\theta)\,d\varphi^2 \tag{13}$$

we obtained exact vacuum solutions through conformal transformations (Askari and Riazi, n.d.).

These solutions, which contain three arbitrary parameters r_0, δ, and B_0, read

$$\frac{r}{r_0} = \frac{2\gamma(\phi(r)/\phi_0)^{\gamma-(\delta+1)/2}}{1 - (\phi(r)/\phi_0)^{2\gamma}} \tag{14}$$

$$B(r) = B_0\left[\frac{\phi(r)}{\phi_0}\right]^{\delta-1} \tag{15}$$

and

$$A(r) = \frac{r^4 B(r)\phi'^2(r)}{Q_s^2} \tag{16}$$

where

$$\gamma^2 = \frac{\omega + 3/2}{2} + \frac{\delta^2}{4} \qquad \text{and} \qquad \phi_0 = \left(\frac{2\omega + 4}{2\omega + 3}\right)\frac{1}{G}$$

G is Newton's gravitational constant. It is straightforward to derive the following expansions for the BD field and metric coefficients from (14)–(16):

$$\begin{aligned}\phi(r) = \phi_0 + \frac{Q_s}{r} &+ \left[1 + \frac{Q_s}{2M_a}\left(\frac{2\omega + 3}{2\omega + 4}\right)\right]\frac{GM_aQ_s}{r^2} \\ &+ \left[\frac{(\omega - 6)Q_s^2}{6M_a^2}\left(\frac{2\omega + 3}{2\omega + 4}\right)^2 + \frac{11Q_s}{3M_a}\left(\frac{2\omega + 3}{2\omega + 4}\right) - \frac{8}{3}\right] \\ &\times \frac{G^2M_a^2Q_s}{2r^3} + \cdots\end{aligned} \tag{17}$$

$$\begin{aligned}B(r) = 1 - \frac{2GM_a}{r} &+ \frac{2G^2M_aQ_s}{r^2}\left(\frac{2\omega + 3}{2\omega + 4}\right) \\ &+ \left[\frac{(\omega - 16)Q_s}{6M_a}\left(\frac{2\omega + 3}{2\omega + 4}\right) + \frac{5}{3}\right]\frac{G^3M_a^2Q_s}{r^3} + \cdots\end{aligned} \tag{18}$$

$$\begin{aligned}A(r) = 1 + \frac{2G}{r}&\left[M_a - \left(\frac{2\omega + 3}{2\omega + 4}\right)Q_s\right] \\ &+ \left[\frac{(8 - \omega)Q_s^2}{2M_a^2}\left(\frac{2\omega + 3}{2\omega + 4}\right)^2 - \frac{9Q_s}{M_a}\left(\frac{2\omega + 3}{2\omega + 4}\right) + 4\right] \\ &\times \frac{G^2M_a^2}{r^2} + \cdots\end{aligned} \tag{19}$$

in which we have defined the new constants of integration M_a and Q_s as

$$2M_a = \left(\frac{2\omega + 3}{2\omega + 4}\right)Q_s(1 - \delta) \qquad \text{and} \qquad Q_s = -\phi_0 r_0$$

and put $B_0 = 1$ (asymptotically Minkowskian metric). We have identified the integration constant M_a as the total active mass of the configuration. This (active) mass naturally includes the mass of the central material body, the gravitational field energy, and the energy of the ϕ-field. The reason for identifying M_a with the active gravitational mass is as follows. The motion of a small test particle is still governed by the geodesic equation in BD theory. The total active gravitational mass (in accordance with its usual definition given in the introduction), is obtained by comparing the weak-field limit of the geodesic equation with Newton's second law.

In fact, the geodesic equation in the form of equation (1) is not rigorously valid in BD theory and even the weak equivalence principle is violated for massive bodies because of the Nordtvedt effect (Nordtvedt, 1968). However, when we are considering the motion of a small laboratory-sized test particle in a weak background gravitational field, deviations from the free-fall geodesic [equation (1)] are of the order of $[1/(2 + \omega)]E_g/M \sim 10^{-39}$, which is incredibly small and therefore negligible (Will, 1981).

Following the conventional method in GR, let us extract from the Einstein tensor a first-order pseudotensor (Weinberg, 1972)

$$G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}R^{(1)\lambda}_{\lambda} \tag{20}$$

$R^{(1)}_{\mu\nu}$ and $R^{(1)\lambda}_{\lambda}$ are computed such as to include only first-order terms in $h_{\mu\nu}$, where

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{21}$$

$\eta_{\mu\nu}$ is the Minkowski tensor (note that $h_{\mu\nu}$ need not be small). The exact BD equations can then be written as

$$G^{(1)\mu\nu} = -\frac{8\pi}{\phi}\,[T^{\mu\nu}_M + T^{\mu\nu}_\phi + t^{\mu\nu}] \tag{22}$$

where $T^{\mu\nu}_M$ is the matter energy-momentum tensor, $T^{\mu\nu}_\phi$ is the ϕ-field energy-momentum tensor, and $t^{\mu\nu}$ is the gravitational field energy-momentum pseudotensor. $G^{(1)}_{\mu\nu}$ obeys the linearized Bianchi identities

$$\frac{\partial G^{(1)\mu\nu}}{\partial x^\mu} = 0 \tag{23}$$

We can therefore conclude that $\tau^{\mu\nu}$ defined according to

$$\tau^{\mu\nu} = \frac{1}{G\phi}\,[T^{\mu\nu}_M + T^{\mu\nu}_\phi + t^{\mu\nu}] \tag{24}$$

is locally conserved

$$\frac{\partial\tau^{\mu\nu}}{\partial x^\mu} = 0 \tag{25}$$

Also, the quantity P^μ defined according to

$$P^\mu = \int_V \tau^{0\mu}\, d^3x \tag{26}$$

is naturally interpreted as the total energy-momentum '4-vector.' It should be remarked that P^λ is not generally covariant, but is a Lorentz vector. It can be shown that P^0 is exactly expressible as (2) with G replaced by $[(2\omega + 4)/(2\omega + 3)]/\phi_0$. Let us see what equation (2) leads to when applied to our solutions of the BD theory. It can be easily shown that

$$h_{ij} = n_i n_j [A(r) - 1] \tag{27}$$

and therefore

$$h_{ij} = \frac{2G}{r}\left[M_a - \left(\frac{2\omega + 3}{2\omega + 4}\right)Q_s\right]n_i n_j + O(r^{-2}) \tag{28}$$

which conforms with h_{ij} of the Schwarzschild metric only in the case $Q_s = 0$.

We can also show that

$$\frac{\partial n_i}{\partial x^j} = \frac{\delta_{ij} - n_i n_j}{r} \tag{29}$$

and

$$\frac{\partial(n_i n_j)}{\partial x^j} = \frac{2n_i}{r} \tag{30}$$

$$\frac{\partial h_{ij}}{\partial x_j} = \frac{2n_i[A(r) - 1]}{r} + n_i\frac{dA(r)}{dr} \tag{31}$$

$$\frac{\partial h_{jj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^j} = -\frac{2n_i[A(r) - 1]}{r} \tag{32}$$

Equation (2) then leads to

$$P^0 = \lim_{r\to\infty} \frac{r}{2G}\,[A(r) - 1] = M_a - \left(\frac{2\omega + 3}{2\omega + 4}\right)Q_s \tag{33}$$

We thus come to the conclusion that P^0 as defined via (28) (which results from the concept of the total mass of a system in special relativity) does not coincide with the usual definition of active gravitational mass unless $Q_s = 0$.

Note that such a case does not arise in GR. As GR is the $\omega \to \infty$, $Q_s \to 0$ limit of BD, $P^0 = M_a$ in GR.

Note that the energy of the ϕ field is contained in both P^0 and M_a, and the difference between these two cannot be assigned to the BD field energy at large distances.

Even when there is a field (like an electric field) which extends to infinity, we still have $P^0 = M_a$ in GR. Consider, for example, the Reissner–Nordström metric, which describes the space-time around a spherically symmetric electric charge q with a total active mass M_a:

$$d\tau^2 = \left(1 - \frac{2GM_a}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{2GM_a}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2$$

As the metric coincides with the Schwarzschild metric up to the first order in $1/r$, we conclude that $P^0 = M_a$, i.e., the same value as the active mass calculated from the geodesic equation.

The concept of the inertial mass of a body as the total energy (including all sorts of it), according to what we have learned from special relativity, encourages us to assess that P^0 might be interpreted as the total inertial mass of the body.

Finally, let us derive an expression which facilitates the computation of total active mass in terms of the matter and ϕ-field energy-momentum tensors. The following identity can be proved easily:

$$G_t^t = \frac{1}{r^2}\left[1 - \frac{d}{dr}\left(\frac{r}{A}\right)\right] \tag{34}$$

We also have from (11)

$$G_t^t = \frac{-8\pi}{\phi}\,(T_{Mt}^t + T_{\phi t}^t) \tag{35}$$

Eliminating G_t^t between (34) and (35) and integrating from $r' = 0$ to r, we obtain

$$\int_0^r \left[1 - \frac{d}{dr'}\left(\frac{r'}{A(r')}\right)\right] dr' = -8\pi \int_0^r \frac{T^t_{Mt}}{\phi} r'^2\, dr' - 8\pi \int_0^r \frac{T^t_{\phi t}}{\phi} r'^2\, dr' \tag{36}$$

A first integration of the left-hand side integral yields

$$\frac{1}{A(r)} = 1 + \frac{8\pi}{r} \int_0^r \frac{T^t_{Mt}}{\phi} r'^2\, dr' + \frac{8\pi}{r} \int_0^r \frac{T^t_{\phi t}}{\phi} r'^2\, dr' \tag{37}$$

The second integral on the right-hand side can be expanded as

$$\int_0^r \frac{T^t_{\phi t}}{\phi} r'^2\, dr' = \int_0^\infty \frac{T^t_{\phi t}}{\phi} r'^2\, dr' + O\left(\frac{1}{r}\right)$$

Comparing the $O(1/r)$ terms in equation (37) with the corresponding term in the expansion for $1/A(r)$ as obtained from (19), we obtain the following relationship:

$$M_a = \frac{2\omega + 3}{2\omega + 4}\left[Q_s - 4\pi \int_0^a \frac{T^t_{Mt}}{(\phi/\phi_0)} r^2\, dr - 4\pi \int_0^\infty \frac{T^t_{\phi t}}{(\phi/\phi_0)} r^2\, dr\right] \tag{38}$$

It can be noted that as $\omega \to \infty$, we have $Q_s \to 0$, $\phi \to \phi_0$, and

$$M_a \to -4\pi \int_0^a T^t_{Mt} r^2\, dr = \int_0^a 4\pi r^2 \rho_{\text{matter}}(r)\, dr \tag{39}$$

which is the usual expression for the total mass (including the gravitational energy) in general relativity.

The exterior metric represented by the expansions (18) and (19) contains two constants of integration M_a and Q_s. We have previously called Q_s the scalar charge of the configuration. What is the physical content of Q_s? This constant has emerged as a result of the first integration of equation (11) in vacuum:

$$r^2\left(\frac{B}{A}\right)^{1/2} \phi' = -Q_s \tag{40}$$

It is related to the energy-momentum of the central matter via

$$Q_s = \frac{-2}{2\omega + 3} \int \sqrt{g}\, T^{\lambda}_{M\lambda}\, d^3x \tag{41}$$

which results from (11).

The important point learned from equation (40) is that the two parameters M_a and Q_s are not really independent. For a configuration of mass M_a, Q_s is determined uniquely by the *state* of matter in the central material sphere. For example, if the central body is made of a traceless energy-momentum tensor $T^{\lambda}_{M\lambda} = 0$, then obviously $Q_s = 0$ by virtue of equation (41) and $M_a = P^0$. On the other hand, it can be shown through the post-Newtonian formalism, or direct calculation (see the Appendix), that for nonrelativistic material ($p << \rho$)

$$\frac{Q_s}{M_a} = \frac{2}{2\omega + 3} \qquad (p << \rho) \tag{42}$$

4. CONCLUSION

Various definitions for the total mass of a spherically symmetric body in general relativity and BD theories of gravitation were discussed. We showed that the conventional definition of the Lorentzian 4-momentum of the body leads to a value for the total mass which is different from the total mass as inferred from the geodesic equation. We conclude that the total active and total inertial masses of a body in BD theory are not necessarily equal. On the other hand, the inequality of passive and inertial masses (violation of the weak equivalence principle) in BD theory is already well known (Nordtvedt effect).

Finally, we obtained an integral representation for the total active mass of a body in BD theory in terms of the energy-momentum tensors of matter and BD scalar field, and discussed the relation between the active mass and the 'scalar charge.'

APPENDIX

We present a direct proof of equation (42) for a perfect, nonrelativistic fluid.

The following identity can be proved easily:

$$\frac{1}{2\sqrt{g}} \frac{\partial}{\partial x^{\rho}} [\sqrt{g}\phi(\ln B)^{\rho}_{,}] = -8\pi T^{t}_{Mt} + (1 + \omega)\, \Box^2\phi \tag{A1}$$

We also have

$$T^{\lambda}_{M\lambda} = T^{t}_{Mt} + T^{i}_{Mi} \simeq T^{t}_{Mt} \qquad (p << \rho) \tag{A2}$$

From the first equation in (11) and (A2), we can obtain

$$-8\pi T^{t}_{Mt} + (1 + \omega)\,\Box^2\phi = -(\omega + 2)\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\rho}}(\sqrt{g}\phi^{\rho}_{,}) \tag{A3}$$

Equation (A1) then leads to

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\rho}}[\sqrt{g}\phi(\ln[B\phi^{2(\omega+2)}])^{\rho}_{,}] = 0 \tag{A4}$$

Integrating this equation over a sphere of radius r and using Gauss' law, we obtain

$$B_{\text{in}}(r) = B_0'\left[\frac{\phi(r)}{\phi_0}\right]^{-2(2+\omega)} \tag{A5}$$

Note that this is an *interior* identity, while a similar equation [equation (15)] holds outside the material sphere.

Matching the interior (A5) and exterior (15) solutions at the boundary of the material sphere, we obtain

$$\delta - 1 = -2(2 + \omega)$$

Now, by using the relation $2M_a = [(2\omega + 3)/(2\omega + 4)]Q_s(1 - \delta)$, we easily derive equation (42).

If we also perform the same calculations for an isothermal sphere ($p = \epsilon\rho$), we obtain

$$\frac{Q_s}{M_a} = \left(\frac{2\omega + 4}{2\omega + 3}\right)\left[\frac{1 - 3\epsilon}{(2\omega + 3) + (\omega + 1)(3\epsilon - 1)}\right]$$

Note that in the limit $\epsilon << 1$, the above equation reduces to equation (42).

ACKNOWLEDGMENT

This publication is contribution 41 of the Biruni Observatory.

REFERENCES

Askari, H. R., and Riazi, N. (n.d.). *Monthly Notices of the Royal Astronomical Society*, submitted.
Brans, C., and Dicke, R. H. (1961). *Physical Review*, **124**, 925.
Bonnor, W. B. (1992). *Classical and Quantum Gravity*, **9**, 269.

Cooperstock, F. I., Sarracino, R. S., and Bayin, S. S. (1981). *Journal of Physics A: Mathematical and General*, **14**, 181.

Herrera, L., and Ibanẽz, J. (1993). *Classical and Quantum Gravity*, **10**, 535.

Nordtvedt, K., Jr. (1968). *Physical Review*, **169**, 1017.

Rosen, N., and Cooperstock, F. I. (1992). *Classical and Quantum Gravity*, **9**, 2657.

Weinberg, S. (1972). *Gravitation and Cosmology*, Wiley, New York.

Will, M. C. (1981). *Theory and Experiments in Gravitational Physics*, Cambridge University Press, Cambridge.